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Non-classical measurement theory: a framework for behavioral sciences

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Non-classical Measurement Theory: a Framework for Behavioral Sciences*

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Abstract

Instances of non-commutativity are pervasive in human behavior. In this paper, we suggest that psychological properties such as attitudes, values, preferences and beliefs may be suitably described in terms of the mathematical formalism of quantum mechanics. We expose the foundations of non-classical measurement theory building on a simple notion of orthospace and ortholattice (logic). Two axioms are formulated and the characteristic state-property duality is derived. A last axiom concerned with the impact of measurements on the state takes us with a leap toward the Hilbert space model of Quantum Mechanics.

An application to behavioral sciences is proposed. First, we suggest an interpretation of the axioms and basic properties for human behavior. Then we explore an application to decision theory in an example of preference reversal. We conclude by formulating basic ingredients of a theory of actualized preferences based in non-classical measurement theory.

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1 Introduction

With Quantum Mechanics (QM) we learned that "strange" (i.e., non-classical) laws rule the world of sub-atomic particles. Recently, the interest for QM has been rapidly expanding. Partly, this is due to the development of quantum computing, which inspires physicists and more recently economists to investigate the use of quantum information in games (Eisert (1999), La Mura (2004)). Another avenue of research has emerged in response to observations that classical (or macro) objects (e.g. human perception or preferences) can exhibit properties specific to QM-objects. In Lambert-Mogiliansky, Zamir and Zwirn (2003)

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a Hilbert space model is proposed to describe economic agents' preferences and decision-making. Aerts (1994), Busemeyer and Townsend (2004) and Khrenikov et al. (2003) investigate quantum-probabilistic like phenomena in psychology. The basic idea is that the mathematical formalism of QM, often referred to as "quantum logic" rather than its physical content, is a suitable model for describing, explaining and predicting human behavioral phenomena in psychology and social sciences.[1]

The term quantum logic appeared first in Birkhoff and von Neumann's seminal article from 1936. They proposed a formulation of the mathematical framework of Quantum Mechanics in terms of a non-classical propositional calculus. The next important contribution is due to Mackey (1963) who provided an axiomatic approach to the probabilistic calculus of standard quantum theory. Piron (1976) significantly extended Mackey's work. His representation theorem and axiomatic framework provided much impetus for further developments. At the same time, Foulis and Randal (1978) developed an approach based on more primitive and operational foundations. For an excellent review of the field see the introductory chapter in Coecke, Moore and Wilce (2000).

A central feature of non-classical (quantum) measurement theory is that performing a measurement on a system affects the system, i.e., it changes its state. As a consequence some experiments or measurements may be "incompatible"; the properties of the system revealed by the measurements depend on the order in which the measurements are performed.

In economics, we use to define an agent by her preferences and beliefs, in psychology by her values, attitudes and feelings. We also talk about "eliciting" or "revealing" preferences and attitudes. This presumes that those properties are sufficiently well-defined (determined) and stable. In particular, it assumes that the mere fact of subjecting a person to an elicitation procedure is not expected to affect her. Yet, psychologists are well aware that simply answering a question about a feeling may affect a person so as to modify subsequently revealed attitudes. For instance when asking a person "Do you feel angry?" a "yes answer" may take her from a blended emotional state to an experience of anger. But before answering the question, it may be neither true nor false that the person is angry. It may be a "jumble of emotions"[27].[2] Similarly, Erev, Bornstein and Wallsten (1993) show in an experiment that simply asking people to state the subjective probability they assign to some event affects the way they make subsequent decisions. The so-called "disjunction effect" (Tversky and Shafir (1992)) may be viewed in this perspective. In a well-known experiment, the authors find that significantly more students report they would buy a non-refundable Hawai vacation if they knew whether they passed the exam or failed compared to when they don't know the outcome of the examination. In the case they passed, some buy the vacation to reward themselves. In the case they failed, some purchase the vacation to console themselves. When they don't know, a seemingly inconsistent behavior is observed: fewer vacations are

[1] The term quantum logic is often used less rigorously to refer to the mathematical formalism of QM. Formally, quantum logic is only about orthomodular lattices however.

[2] The distinction between the two states before and after she answers the question, can be evidenced in her answers to subsequent questions.

being purchased than in any one of the two possible events. We may say that the students who don't know are in a blended state of mind: neither satisfied nor disappointed. Non-classical measurement theory has developed a notion to capture uncertainty not due to incomplete information (represented as a mixture), the notion of "superposition". The data in the Tversky-Shafir experiment are consistent with the expected behavior of students who are in a superposed state. Cognitive dissonance phenomena are another instance of seemingly inconsistent behavior that can be explained in terms of quantum indeterminacy.[3]

In contrast with Physics, we do not in behavioral sciences have precise experimental evidence, which creates an absolute necessity for rejecting the classical model.[4] Yet, there is a host of experimental phenomena that standard theory cannot explain without additional and some time fairly ad-hoc assumptions (see Kahneman and Tversky (2000), Camerer 2003). Interestingly, Kahneman and A. Tversky explicitly discuss some anomalies in terms of measurement theory: "*Analogously, - to classical physical measurement - the classical theory of preference assumes that each individual has a well-defined preference order and that different methods of elicitation produce the same ordering of options". But, "In these situations - of violation of procedural invariance - observed preferences are not simply read off from some master list; they are actually constructed in the elicitation process.*" ([11] p. 504). Nobel Prize laureate A. Sen (1997) also emphasizes that the "act of choice" has implications for preferences. These and other "intriguing analogies" between QM phenomena and psychological and behavioral phenomena suggest that non-classical measurement theory can be useful for the description and the modeling of human behavior.[5]

The objective of this paper is to provide an exposition of the foundations of non-classical measurement theory allowing to assess its relevance for behavioral and social sciences. We find that its axioms and properties can be formulated so as to allow for a meaningful interpretation. The non-classical man who emerges from our investigation is a structurally "plastic" creature. This is due to a property of non-orthogonality (connectedness) of behavioral (pure) states. This central feature of the theory implies an irreducible uncertainty in behavior, which is also the source of change. When the non-classical man interacts with his environment, e.g., makes a choice in a given decision context, some uncertainty is resolved which prompts a modification of his behavioral state. The non-classical man is essentially contextual. In a simple example, we apply the model to choice behavior to show that non-classical measurement theory can explain preference reversal phenomena. Finally, we suggest how the framework could be used to develop a theory of actualized preferences.

Section 2 offers a few examples of quantum and quantum-like behavior. In section 3 we informally

[3]For a HSM (Hilbert Space Model) of cognitive dissonance see [14].

[4]Of course, not all instances of non-commutativity in behavior call for a questioning the classical approach at all. In standard consumer theory, choices do have implications for future behavior, i.e., when goods are substitutes or complements. Our focus is on the so-called "behavioral anomalies", i.e., on cases when nothing justifies a modification of preferences (or beliefs). Yet, actual behavior reveals a change.

[5]D. Luce (2005) also writes about analogies between physical and behavioral measurements. Luce deals with classical measurements and his focus is on similarities in terms of the functional forms.

introduce some basic notions of non-classical measurement theory. The formal framework is exposed in section 4 and 5. Section 6 discusses an interpretation of the basic axioms and properties for behavioral sciences, and develops an application in a decision theoretical example.

2 Examples

Example 1: *The spin of an electron*

An electron is endowed with several properties including mass, charge and spin. To fix idea one can imagine the electron as a cricket ball that rotates around an axis. Since it is charged, this rotation affects its magnetic moment, the spin, which can be measured.[6]

A first result is that the outcome of measurement is always ± one and the same magnitude independently of the orientation of the measurement device. If we measure a concrete electron along some axis x and obtain result $+$, then a new measurement along the same axis will give the same result. Assume we prepare a number of electrons this way. If we for the second measurement modify the orientation of the axis e.g. the measurement device is turned by $90°$, the result now shows equal probability for both outcomes. As we anew perform the measurement along the x-axis, we do not recover our initial result. Instead, the outcome will be - with .5 probability.

For our purpose, we only note that once the spin of the electron along some axis is known, the results of the measurement of the spin of that electron along some other axis has a probabilistic character. This is a central feature. In the classic world, we are used to deal with probabilities. But there the explanation for the random character of the outcome is easily found. We simply do not know the exact state of the system, which we represent by a probability mixture of other states. If we sort out this mixture in the end we obtain a pure state and then the answer will be determinate. In the case with the spin, it is not possible to simultaneously eliminate randomness in the outcome of measurements relative to different axis.

Example 2: *A fly in a box*

Consider a box divided by two baffles into four rooms (left/front (LF), Left/Back (LB), RF and RB. In this box, we hold a fly that flies around. Because of the baffles, it is limited in its movements to the room where it is.

Assume that we only have access to two types of measurements. The first allows answering the question whether the fly is in the Left (L) or the Right (R) half of the box. And, in the process of measurement the baffles between the Front (F) and the Back (B) half of the box is lifted while the separation between Right and Left is left in place. During that process, the fly flies back and forth from Front to Back. When the measurement operation is over and the baffle between Front and Back put

[6]Stern and Gerlagh created an instrument such that the interaction between the magnetic moment of the electron and that of the experimental setup generates the splitting of a beam of electrons. A measure of the deviation can be interpreted of the measurement of spin (along some orientation).

back in place, the position of the fly is therefore quite random (LF or LB). The same applies for the measurement of Front/Back.

Assume that we have performed the measurement L/R and obtained answer L. Repeating that same measurement even 100 times we will always obtain the same answer L. But if we do, in between, make the F/B measurement, we have equal (for the sake of simplicity) chance to obtain R as L. We see that the behavior of our system reminds of that of the spin (when the Stern Gerlach device is rotated by an angle of 90°). Here the position of the fly cannot be determined with certainty with respect to the two measurements (LR) and (FB) simultaneously. The measurement affects the system in an uncontrollable and unavoidable way.[7]

Example 3: *Attitudes and values*

Consider the following situation. We are dealing with a group of individuals and we are interested in their preferences (or attitudes). We dispose of two tests.

The first test is a questionnaire corresponding to a Prisoners' Dilemma like situation with options C and D, against an anonymous opponent. The second test corresponds to the first mover's choice in an Ultimatum Game (UG). The choice is between making an offer of (9,1) or of (4,6).[8]

The observations we are about to describe cannot obtain in a world of rational agents whose preferences are fully described by their monetary payoff.[9] But this is not our point. Our point is that such observations exhibit the same patterns as the ones we described in the spin and fly example above.[10]

Suppose that we have the following observations. The respondents who answer C to the first questionnaire when asked immediately again they repeat (with probability close to one) their answer. We now perform the second test (UG) and the first test (PD) again. In that last PD test we observe that not all respondents repeat their initial answers. A (significant) share of those who previously chose to cooperate now chooses to defect.

What is going on? When deciding in the PD our respondent may feel conflicted: she wants to give trust and encourage cooperation, but she does not like to be taken advantage of. Consider the case when

[7]This simple example exhibits some (but not all) of the basic features of the non-classical measurement theory developed in this paper.

[8]In the Ultimatum game the first mover makes an offer. The respondent either accepts the deal and the payoffs are distributed accordingly. Or he refuses in which case no one receives any payoff.

[9]We know that game theory uniquely predicts behavior: people defect (D) in the PD and with common knowledge of rationality, they offer (9,1) in UG. Experimentalists have however taught us to distinguish between monetary payoffs, which can be controlled and preferences, which may include features beside monetary payoffs unknown to the designer of the experiment.

[10]D. Balkenberg and T. Kapplan (University of Exeter, unpublished) conducted an experiment with those two same games but with two populations of respondents. They investigate the frequency of the choices when the two games are played in one order compared to when they are played in the reverse order. The data shows an impact of the first choice on the second characteristic of non-classical measurements.

her optimistic 'I' takes over: she decides to cooperate. When asked again immediately after, her state of mind is that of the optimistic 'I' so she feels no conflict: she confirms her first choice. Now she considers the UG. The deal (4,6) is very generous but it may be perceived as plain stupid. The (9,1) offer is not generous but given the alternative it should not be perceived as insulting. She feels conflicted again because her optimistic 'I' does not provide clear guidance. Assume she chooses (9,1). Now considering the Prisoners' Dilemma again, she feels conflicted anew. Indeed, her choice of (9,1) is not in line with the earlier optimistic mood so she may now choose to defect.

As in the spin and the fly example, the measurement (of attitudes) affects the agent (system) in an uncontrollable way so observed behavior (measurement outcomes) may exhibit instances of non-commutativity characteristic of quantum measurement theory.

3 Non-classical measurements

In this section, we propose a non-formal introduction to some basic concepts. The formal definitions are given in the next section where the theory is developed.

Suppose that we are dealing with some system (an electron, the taste template of a person, a fly in a box). In each moment the system is characterized by its state, which encapsulates all information about the system. We dispose of various experimental set-ups to perform measurements on that system. Our system may give different answers to one and the same measurement when it is in different states. The state is the main (and ideally the single) reason for the differences between the answers. Our system in one and the same state may give different answers to one and the same question however. In the classical world the probabilistic character of outcomes can be explained by our incomplete knowledge of the true or pure state.[11] Non-classical measurement theory allows for an intrinsic (objective) probabilistic character of measurement outcomes i.e. for an uncertainty not due to our incomplete knowledge. In the following the notion of state is used to refer to *pure states* i.e., states that are not (non-trivial) probabilistic mixtures of other states.

An experimental set-up (ES) is concretely made of electrical wires, magnets, a questionnaire, a decision situation, which can be difficult to fully describe. However, all we need to know is the set of possible outcomes denoted $O(M)$ associated with the experimental set-up M. For instance, in the example with the spin the outcomes are + or - for a given orientation in space. In the fly or decision example, we had two experiments each with a set of outcomes including 2 elements. Generally, the outcomes of a measurement may be elements of an arbitrary abstract set. In quantum mechanics, the outcomes are real numbers, i.e., the eigenvalues of the Hermitian operator (called "observable") that represents the property that is being measured. In our context, the results of a measurement are not numbers, they are conventional "labels". As a special case, we may also encounter numbers.

[11] Incomplete information implies that the state is a probability mixture of other states. Using measurements, we can sort out or "filter" this mixture so as to eventually obtain a pure state which we fully know.

When the system *interacts* with the experimental set-up M, we obtain as a result one of the values belonging to $O(M)$. We say that the concrete value $o \in O(M)$ is the *outcome of (performing) the measurement.*

Measurements constitute a special class of interactions. We focus on non-destructive measurements, which means that the system is not destroyed in the process of measurement (but the state of the system may change). This implies that we can perform new measurements on the system. The most important feature characterizing a measurement is that if we perform the *same measurement* (of the same system) twice in a row, we obtain the *same answer.* This property is referred to as von Neumann's "Measurement Postulate". Measurements satisfying this postulate are called "first-kind measurements".[12]

The measurement postulate assumes that the system does not have an own dynamic (or that the measurements can be done sufficiently close in time so as to prevent it from evolving). Measuring the position of a running train or asking a drug addict about his willingness to pay for heroin, does not qualify as a first-kind measurement. Another example of a measurement that is not first-kind, is when we pick up an individual from a population and ask him a question. Here, the system is the population. If we repeat this measurement by picking up another individual, the answer will be different from the first one. Our system must in some sense be elementary.

First-kind measurements invite an interpretation according to which the outcome of a measurement provides information about a *property* of the system. If we measure the color of a system, obtain "red" and, when we repeat the experiment we obtain red again then we are inclined to think that the system really is of the red color and that our measurement only reveals a property that pre-existed the measurement. We simply did not know about it. Our approach consists in proposing that the property "be red" may not have pre-existed the measurement. Instead, it obtains as a result of the interaction with the experimental set-up. We shall also say that the property is *actualized* (rather than revealed) by the experimental set-up. The property truly pertains to the system, which can be confirmed by a repeating the measurement. But it can disappear, if we perform the measurement of another property that is *incompatible* (see below) with the one measured first.

[12]There is no definite example of a physical quantity not admitting at least one measuring instrument satisfying, even approximately, the von Neuman and the more requiring Luder's postulate (applying to coarse measurements). The instruments used in reality may not satisfy the postulates however.

Figure 1 provides an illustration of first-kind non-classical measurements with two outcomes.

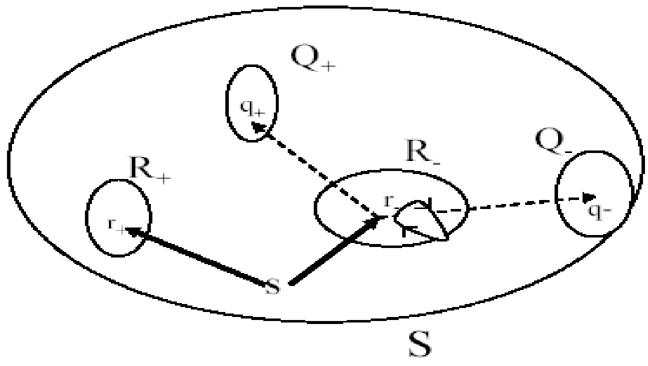

Figure 1

The state of the system before the measurement is represented by point s. Any state $s \in \mathbb{S}$, where $\mathbb{S}$ is the set of pure states, if measured will yield outcome R_+ with some probability and R_- with the complementary probability. When we perform the measurement, the state s jumps either in R_+ and gives the outcome R_+ or in R_- and gives the outcome R_-. One says that the state s is a *superposition* of some state $r_+ \in R(+)$ and $r_- \in R(-)$. When $s \in R_+$ (after a measurement of R) and we perform this measurement again, we obtain the same answer R_+ with certainty, which is illustrated by the loop in subset R_+.

To precise the intuition, we now compare the non-classical measurement of figure 1 with a classical measurement in figure 2

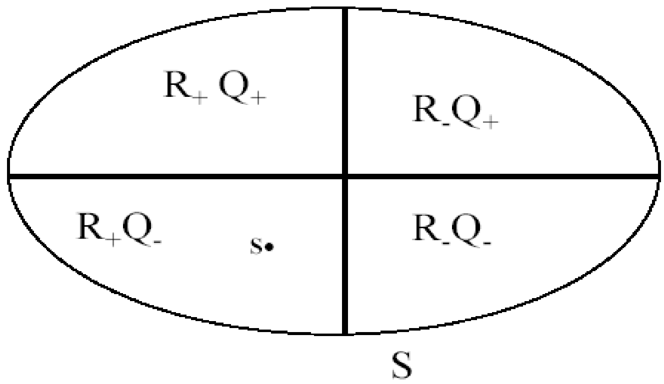

Figure 2

The classical measurement R is represented by a partition of the state space: $R_+ \cup R_- = \mathbb{S}$. The true state of the system is represented by the point s. There is no effect of the measurement of R on the state. It (only) reveals that $s \in R_+$.

Figure 1 and 2 can be used to illustrate one more important notion: that of incompatible properties or non-commuting measurements. Suppose that the measurement of R yielded r_+ and that we thereafter performed a measurement of Q so the state is now either some $q_+ \in Q_+$ or $q_- \in Q_-$, $q_+, q_- \neq p_+$. When performing R once more the state of the system will change again. With some positive probability the system transits to state r_-, i.e., we do not recover the initial result R_+. In contrast, in the classical model measurements always commute with each other. The outcome of a measurement of R always is R_+ whether Q has been measured or not.

We next proceed with the description of the formal framework. Our objective is to expose the basic structure in a simple way. To this end we have divided the presentation into two parts. The first is concerned with the basic set theoretical structure. This structure only provides a rather general measurement theory. The second part focuses on the dynamics of measurement. We there introduce a last axiom that takes us with a leap toward the fully structured Hilbert space model of Quantum Mechanics.

4 States and Properties

4.1 Measurements

Our starting point is an experimental (or measurement) set-up M. $O(M)$ denotes the set of outcomes of measurement M. We assume for simplicity that $O(M)$ is a finite set. $\mathbb{S}$ denotes the set of states of our system. The results from performing measurement M is described by the mapping

$$\mu = \mu_M : \mathbb{S} \to \Delta\left(O\left(M\right)\right). \tag{1}$$

Here $\Delta(X)$ denotes the simplex of probabilistic measures on a finite set X. So for any state $s \in \mathbb{S}$, and outcome $o \in O\left(M\right)$, the non-negative *number* $\mu_M\left(o|\, s\right)$ is the probability that the outcome o is observed when performing measurement M on the system in state s. We assume that for any outcome $o \in O(M)$ there exists a state $s \in S$ such that $\mu_M(o|s) > 0$. If A is a subset in $O(M)$, we write $\mu_M(A|s) := \sum_{a\in A} \mu_M(a|s)$.

Definition 1 *A measurement M is a* first-kind measurement *if the following property is satisfied. Suppose we performed the measurement M on a system and obtained an outcome o. If we perform M on the same system immediately after*[13] *we will obtain the same outcome o with probability 1.*

For $A \subset O(M)$, $E_M(A)$ denotes the set of states s such that $\mu_M(A|s) = 1$. In particular, the set $E_M(o)$ for $o \in O(M)$ is called an *eigenset* of M corresponding to the outcome o. The eigensets $E_M(o)$ are nonempty. Indeed, the outcome o occurs with positive probability; after obtaining outcome o the new state is in $E_M(o)$ due to the first-kindness of M. A measurement M is *complete* if all its eigensets are singletons. While a measurement with two outcomes only (conventionally denoted YES and NO) is called a *question.*

[13] "Immediately after" means that no other measurement is performed between the two M measurements.

$\mathcal{M}$ denote the set of all (admissible) first-kind measurements. We assume that different measurements are interconnected by a monotonicity postulate. To formulate it we introduce the notion of *property.*

Definition 2 *A subset $P \subset \mathbb{S}$ is called a (testable)* property *if $P = E_M(A)$ for some measurement $M \in \mathcal{M}$ and some $A \subset O(M)$.*

Monotonicity postulate. Suppose that $E_M(A) \subset E_{M'}(A')$ for two measurements M, M' and $A \subset O(M)$, $A' \subset O(M')$. Then for any state s the following inequality holds $\mu_M(A|s) \leq \mu_{M'}(A'|s)$.

In particular, if $P = E_M(A) = E_{M'}(A')$ is a property then the probabilities $\mu_M(A|s)$ and $\mu_{M'}(A'|s)$ are equal and depend only on the property P. Therefore we can denote this number as $s(P)$ and understand it as the probability to obtain the property P when performing a measurement of the system in state s. The number $s(P)$ is well-defined and does not depend on the specific measurement. Of course,

$$s(P) = 1 \Leftrightarrow s \in P.$$

The set of all properties is denoted by $\mathcal{P}$. As a subset of $2^{\mathbb{S}}$, $\mathcal{P}$ is a poset (a partially ordered set). It has the minimal element $\mathbf{0} = \emptyset$ and the maximal element $\mathbf{1} = \mathbb{S}$. Any state $s \in \mathbb{S}$ defines the monotone function $s : \mathcal{P} \to \mathbb{R}_+$, $s(\mathbf{0}) = 0$, $s(\mathbf{1}) = 1$. The value $s(P)$ is the probability that the system transits from the state s into the subset P after the performance of an appropriate measurement.

We now describe another basic structure of the property poset $\mathcal{P}$. Let $P = E_M(A)$ be a property. If $\bar{A} = O(M) - A$ is the complement to A, then the subset $P' = E_M(\bar{A})$ is a property too. We have $s(P') = 1 - s(P)$ for any state $s \in \mathbb{S}$. Therefore

$$P' = \{s \in \mathbb{S}, s(P') = 1\} = \{s, s(P) = 0\}$$

and the set-property P' depends only on P. We call it the *opposite* property to P. It is obvious that $P \vee P' = \mathbf{1}$ in the poset of properties $\mathcal{P}$. In other words, $\mathcal{P}$ is an "ortho-poset".

4.2 Orthogonality

Let us now introduce the basic structure of the state space: orthogonality. We say that two states s and t are *orthogonal* (and write $s \perp t$) if there exists a property P such that $s(P) = 1$ and $t(P) = 0$. Since for the opposite property P', it holds that $s(P') = 0$ and $t(P') = 1$, we have $t \perp s$, so that $\perp$ is a symmetric relation on the set $\mathbb{S}$. Obviously, $\perp$ is an irreflexive relation.

Definition 3 *An* orthogonality relation *on a set X is a symmetric and irreflexive binary relation $\perp \subset X \times X$. A set X equipped with an orthogonality relation $\perp$ is called an* orthospace.

Consider, for example, the Euclidian space H equipped with a scalar product (x, y). We say that vectors x and y are orthogonal if $(x, y) = 0$. The symmetry of the orthogonality relation follows from the symmetry of the scalar product. To obtain the irreflexivity we have to remove the null vector. So that $X = H\backslash\{0\}$ is an orthospace. A similar example provides a Hilbert space over the field of complex numbers.

When graphically representing an orthospace, we may connect orthogonal elements, we then obtain a (non-oriented) graph. This representation is often the most "economic" i.e., few edges need to be written. Or we may, more intuitively, connect non-orthogonal or *tolerant*[14](we also use the term "connected") elements. The graph of tolerantness quickly becomes extremely complex however. In simple cases, we combine the two representations. Dotted lines depict orthogonality and solid lines depict tolerantness. The graphs of example 2 (A fly in a box) is in figure 4.

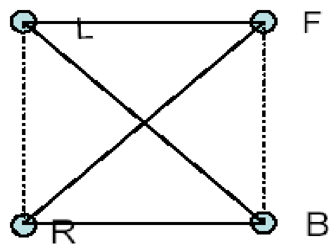

Figure 3

For $A \subset X$ we denote $A^{\perp}$ the set of all states, orthogonal to all elements of A,

$$A^{\perp} = \{x \in X, x \perp A\}.$$

For instance $\emptyset^{\perp} = X$ and $X^{\perp} = \emptyset$. If $A \subset B$ then $B^{\perp} \subset A^{\perp}$.

Definition 4 *The set* $A^{\perp\perp}$ *is called the* ortho-closure *of a subset* A. *A set* F *is said to be* ortho-closed *(also called a* flat *) if* $F = F^{\perp\perp}$.

It is easily seen that for any $A \subset X$, the set $A^{\perp}$ is ortho-closed. Let $F = A^{\perp}$, clearly $F \subset F^{\perp\perp}$. On the other side $F^{\perp} = A^{\perp\perp} \supset A$; applying $\perp$ we reverse the inclusion relation $F^{\perp\perp} \subset A^{\perp} = F$. In particular, the ortho-closure of any A is ortho-closed.

$\mathcal{F}(X, \perp)$ denotes the set of all flats of X ordered by set theoretical inclusion, which we denote $\leq$. It contains the largest element X, denoted **1**, and the smallest element $\emptyset$ denoted **0**. Moreover the poset $\mathcal{F}(X, \perp)$ is a (complete) lattice. The intersection of two (or more) flats is a flat implying that $A \wedge B$ exists and equals $A \cap B$. The join $A \vee B$ also exists and is given by the formula

$$A \vee B = (A \cup B)^{\perp\perp}.$$

[14]The term "tolerant" is used in mathematics to refer to a symmetric and reflexive relation.

Definition 5 *An ortholattice is a lattice equipped with a mapping* $\perp\colon \mathcal{F} \to \mathcal{F}$ *such that*

i. $x = x^{\perp\perp}$;

ii. $x \le y$ *iff* $y^{\perp} \le x^{\perp}$;

iii. $x \vee x^{\perp} = \mathbf{1}$.

So that the poset $\mathcal{F}(X,\perp)$ is an ortholattice. The ortholattice $\mathcal{F}(\mathbb{S},\perp)$ can be called the *logic* of the system.

4.3 The state-property duality

We assume further that the following axiom is fulfilled:

Axiom 1 *For any state* $s \in \mathbb{S}$, *the set* $\{s\}$ *is a property.*

In other words, there exists a measurement such that $\{s\}$ is one of its eigensets. This is a crucial and substantive assumption. It means that given any state, there exists an experimental set-up which can "prepare" the system in that state. Axiom 1 implies that the ortho-poset of properties $\mathbb{P}$ is atomistic.

Let s and t be two states. Due to Axiom 1, we can speak about $s(t) := s(\{t\})$, the probability for a transition from the state s to the state t. In general case $s(t)$ can differ from $t(s)$ (see Example 2, section 2). But the following weaker property holds

$$s(t) = 0 \text{ iff } t(s) = 0.$$

It is a consequence of the following

Lemma 1 $t(s) = 0$ *iff* $s \perp t$.

Proof. Let P be a property with $s(E) = 1$ and $t(E) = 0$. Then $s \in P$. Therefore $P' \subset \{s\}'$ and $t(P') = 1 - t(P) = 1$; by monotonicity we have $t(\{s\}') = 1$ and consequently $t(s) = 0$. The opposite assertion is obvious. ■

Axiom 1 permits to consider the set $\mathbb{S}$ as the set of PURE states. That is, any state s considered as a function on the poset $\mathcal{P}$ is not a mixture (a convex combination) of other states. Indeed, suppose that a state s is a convex combination, $s = \sum_i \alpha_i s_i$, where $s_i \in \mathbb{S}$, $\alpha_i > 0$ and $\sum_i \alpha_i = 1$. Let us consider the value of the function $s = \sum_i \alpha_i s_i$ in the state-property $\{s\}$. Since $s(\{s\}) = 1$, we obtain $1 = \sum \alpha_i s_i(s)$. On the other side, $s_i(s) \le 1$. Therefore $s_i(s) = 1$ for any i, that is $s_i = s$ for any i.

Another consequence of Axiom 1 is the following

Proposition 1 *Any property is a flat.*

Proof. Let P be a property. We assert that $P = P'^{\perp}$, where P' is the opposite property. The inclusion $P \subset P'^{\perp}$ is obvious. Let us check the inverse inclusion. Suppose that a state s is orthogonal to P'. Then

$P' \subset \{s\}^{\perp}$. Since $s(\{s\}^{\perp}) = 0$ we obtain from monotonicity that $s(P') = 0$ and $s(P) = 1 - s(P') = 1$ that is $s \in P$. ■

Note that we now have that $P' = P^{\perp}$. Therefore the natural inclusion of $\mathcal{P}$ in $\mathcal{F}(\mathbb{S}, \perp)$ is a morphism of ortho-posets. (Axiom 2 below yields an identification of these two posets.) Moreover, this inclusion is tight in the sense that any flat F is the meet (or intersection) of some properties. Indeed, $F = \cap\{s\}^{\perp}$, where s runs over the set $F^{\perp}$. In particular, the ortho-closure $A^{\perp\perp}$ of a set A is the set of states s characterized exclusively by the properties common to all states of A. For this reason, one can say that the elements of the ortho-closure $A^{\perp\perp}$ are *superpositions* of A.

Another consequence of Axiom 1 is that the orthospace $\mathbb{S}$ is *ortho-separable*. An orthospace $(X, \perp)$ is called ortho-separable if any one-element subset x of X is a flat. It is easy to see that x is a flat iff for any $y \neq x$ there exists z orthogonal to x but not to y. For example, the orthospace in figure 4

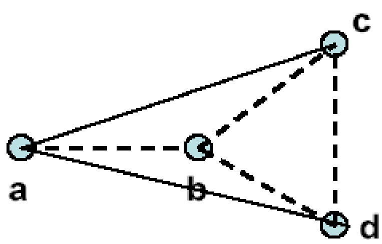

Figure 4

is not ortho-separable, since the closure of $a^{\perp\perp} = \{a, c, d\} \neq a$.

We note that the ortho-separable orthospace $(X, \perp)$ can be reconstructed from the ortholattice $\mathcal{F}(X, \perp)$. Recall that an *atom* of a lattice $\mathcal{F}$ is a minimal non-zero elements of F. A lattice $\mathcal{F}$ is called *atomistic* if any element of the lattice is the join of atoms. If $(X, \perp)$ is an ortho-separable orthospace then $\mathcal{F}(X, \perp)$ is a complete atomistic ortho-lattice.

Conversely, let $\mathcal{F}$ be a complete atomistic ortho-lattice. Then there exists an ortho-separable orthospace $(X, \perp)$ (unique up to isomorphism) such that $\mathcal{F}$ is isomorphic to $\mathcal{F}(X, \perp)$. As X one needs to take the set of atoms of $\mathcal{F}$; atoms x and y are orthogonal if $x \leq y^{\perp}$. More in detail see, for example, [21]. Roughly speaking, orthospaces and ortho-lattices are equivalent objects. Under axiom 1 and 2 (below) this abstract equivalence translates into the state-property duality characteristic of non-classical measurement theory.

4.4 Orthomodularity

Axiom 1 yields that any property is a flat. Here we shall introduce Axiom 2 which implies the inverse.

Suppose that $M \in \mathcal{M}$ is a measurement. Due to Proposition 1, every eigenset $E_M(o)$, $o \in O(M)$, is a flat. Moreover, different flats $E_M(o)$ are orthogonal to each other. Finally, the join of all $E_M(o)$

is equal to $\mathbb{S} = \mathbf{1}$. Indeed, if a state s is orthogonal to $E_M(o)$ then $E_M(o) \subset s^\perp$ and consequently $s(E_M(o)) \leq s(s^\perp) = 1 - s(s) = 0$; on the other hand, $\sum_o s(E_M(o)) = 1$.

Definition 6 *We called an* Orthogonal Decomposition of the Unit (ODU) *a finite family of flats* $(F_i,\ i \in I)$*, such that*

a. any F_i *and* F_j $(i, j \in I,\ i \neq j)$ *are orthogonal to each other;*

b. $\vee_{i \in I} F_i = \mathbf{1}$.

So for a measurement M, the family of flats $(E_M(o),\ o \in O(M))$ is an ODU. The reciprocal of this assertion is assumed in Axiom 2.

Axiom 2 *For any ODU* $(F_i,\ i \in I)$*, there exists a measurement* $M \in \mathcal{M}$ *with the outcome set* $O(M) = I$ *and the eigensets* $E_M(i) = F_i$ *for* $i \in I$.

Proposition 2 *Under Axiom 2 any flat is a property.*

Proof. Let F be a flat. Since the pair $(F, F^\perp)$ is an ODU then by Axiom 2 F is the eigenset of the corresponding measurement-question. ■

In particular, for any flat F and any state s, the number $s(F)$ is defined.

Proposition 3 *Let* $(F_i,\ i = 1, ..., n)$ *be a finite family of mutually orthogonal flats. Then, for any state* s*, it holds that*

$$s(F_1 \vee ... \vee F_n) = s(F_1) + ... + s(F_n).$$

In other words, any state considered as a function on the ortho-lattice $\mathcal{P} = \mathcal{F}(\mathbb{S}, \perp)$ is a probability measure.

Proof. Denote $F = F_1 \vee ... \vee F_n$ and $F_0 = F^\perp$. Then the family $(F_0, F_1, ..., F_n)$ is an ODU. Due to Axiom 2, there exists a measurement with corresponding eigensets. Therefore $s(F_0) + s(F_1) + ... + s(F_n) = 1$. Similarly $s(F) + s(F^\perp) = 1$, that is $s(F) = 1 - s(F_0)$, from which we obtain $s(F) = s(F_1) + ... + s(F_n)$. ■

Another important structural consequence of Axiom 2 is the *orthomodularity* of the lattice of flats. It was early recognized that the failure of classical logic to accommodate quantum phenomena was due to the requirement that the lattice of properties should satisfy the law of distributivity. Birkhoff and von Neumann proposed to use the modularity law. However, the notion of orthomodularity proved to be more adequate, see [12].

Definition 7 *The ortholattice* $\mathcal{F}$ *is orthomodular if* $b = a \vee (b \wedge a^\perp)$ *for any* $a, b \in \mathcal{F}$ *such that* $a \leq b$.

Under Axiom 2, we have the following result

Theorem 1 *The logic* $\mathcal{F}(\mathbb{S}, \perp)$ *is orthomodular ortholattice.*

Proof. Let F and G be two flats and $F \subset G$. Let us consider $H = (F \vee G^{\perp})^{\perp} = F^{\perp} \wedge G$. We assert that $F \vee H = G$. The inclusion $F \vee H \subset G$ is trivial and we only need to establish the reverse inclusion. Suppose that a state $s \in G$, that is $s(G^{\perp}) = 0$. The key remark is that the triplet of flat $(F, H, G^{\perp})$ forms an ODU. Therefore $s(F) + s(H) + s(G^{\perp}) = 1$, $s(F) + s(H) = s(F \vee H) = 1$ and $s \in F \vee H$. That is $G = F \vee (G \wedge F^{\perp})$. ∎

4.5 Decomposition of a system

Definition 8 *A flat $C \subset \mathbb{S}$ is classical (or central) if its set-theoretic complement $\mathbb{S} \setminus C$ is a flat too.*

In other words, C and $\mathbb{S} \setminus C$ are orthogonal to each other. The subsets $\mathbb{S}$ and $\emptyset$ are the *trivial* central flats.

If all flats are classical one can say that the system is classical. In this case all states are orthogonal to each other, and the lattice $\mathcal{F}(\mathbb{S}, \perp)$ is a Boolean one.

Suppose we have two complementary flats C_1 and C_2. We claim that states from different C_i cannot form non-trivial superpositions. Indeed, let $s_1 \in C_1$, $s_2 \in C_2$, and let s be a superposition of s_1 and s_2 different from s_1 and s_2. Since $C_1 \cup C_2 = \mathbb{S}$, the state s is either in C_1 or in C_2; suppose that $s \in C_1$. Due to ortho-separability of $\mathbb{S}$, there exists a state t which is orthogonal to s_1 and is not orthogonal to s. This t must be in C_1. But in this case t is orthogonal to s_2 too. We see thereby that t is orthogonal to the both s_1 and s_2 and consequently it is orthogonal to its superposition s. A contradiction!

It is easy to see that the intersection (or the union) of any number of central flats is a central flat. We say that states s and s' are equivalent ($s \approx s'$) if there is no central flat that contains s and does not contain s', i.e., the two states are characterized by the same classical properties. The relation $\approx$ is an equivalence relation indeed. Let $\Omega = \mathbb{S}/\approx$ be the corresponding coset space. Elements ω of Ω can be considered as classical superstates of our system $\mathbb{S}$.

For $\omega \in \Omega$ denote by $\mathbb{S}_{\omega}$ the corresponding equivalence class in $\mathbb{S}$ with the induced orthogonality. The orthospace $\mathbb{S}_{\omega}$ is *irreducible* in the following sense

Definition 9 *An orthospace $\mathbb{S}$ is irreducible (or fully non-classical, or connected) if it has only trivial central flats.*

If F is a flat in $\mathbb{S}$, then the intersection $F \cap \mathbb{S}_{\omega}$ is a flat in $\mathbb{S}_{\omega}$. Conversely, suppose we have flats F_{ω} in $\mathbb{S}_{\omega}$ for every ω. Then its union $F = \cup_{\omega} F_{\omega}$ is a flat in $\mathbb{S}$. In other words, any system is the direct (orthogonal) sum of irreducible subsystems. Thus, we can restrict ourselves to irreducible systems (or orthospaces.).

There exists a simple criterion of irreducibility.

Proposition 4 *An orthospace $\mathbb{S}$ is irreducible iff any two states can be connected by a chain of tolerant states.*

A system is said to be purely non-classical if the corresponding orthospace of state S is irreducible (and contains more than one element).

5 The Impact of Measurement

Up to now we have focused on the basic set theoretical structure of non-classical measurement theory. We now turn to the elements of the theory dealing with its dynamics, i.e. with the impact of performing a measurement on the state.

5.1 Ideal measurements

Suppose that our system is not purely classical. Then there exists a flat-property F such that $F \cup F^{\perp}$ differs of the whole space $\mathbb{S}$. Suppose that we perform a measurement-question Q corresponding to the pair of flats $(F, F^{\perp})$. If a state s is not in $F \cup F^{\perp}$, it jumps into F (with probability $s(F)$) or into $F^{\perp}$ (with the complementary probability $s(F^{\perp})$). Indeed, suppose that the outcome was YES; due to the first-kindness of the measurement, performing Q again and again will always give the outcome YES with certainty, that is the new state s' lies in the eigenset F. Similarly in the case of the outcome NO.

Thus, in general, measurements change the state. Assume that, as the result of performing measurement M, the system's state transits from s to a new state $s' \in F$. Can we say anything more about the state s'? To this end we introduce a notion of ideal question.

Let $Q \in \mathcal{M}$ be a question with eigensets F and $F^{\perp}$. A property P is *compatible* with the question Q if $F \subset P$ or $F^{\perp} \subset P$. A question Q is called *ideal* if it conserves any property P compatible with Q.

In other words, if a state s is in P before performing Q (where P is supposed to be compatible with Q) then, after performing Q, the state s' is in P too. In some sense an ideal question minimally impacts on states, that is it produces "a least perturbation". Suppose, for example, that s belongs to the eigenset F of the ideal question Q; then under the impact of Q the state s does not change at all.

More generally, let P be the flat $s \vee F^{\perp}$. P is compatible with question Q. Therefore, after performing measurement Q and having obtained answer YES the resulting state s' lies in F (due to the first-kindness of Q) and in P (due to the ideality of Q). That is s' is in the flat $F \wedge (s \vee F^{\perp})$. Of course, in the case of answer NO the state s' is in the flat $F^{\perp} \wedge (s \vee F)$.

We see that the impact of an ideal question is related to the corresponding *Sasaki projection*. Let $\mathcal{F}$ be an ortholattice and let $a \in \mathcal{F}$. The Sasaki projection is a mapping $\varphi_a : \mathcal{F} \to \mathcal{F}$ given by the formula

$$\varphi_a(b) = a \wedge (b \vee a^{\perp}).$$

So, when a system initially in state s gives answer YES to ideal question Q, the state s "jumps" into a state s' belonging to the flat $\varphi_F(s) = F \wedge (s \vee F^\perp)$.

We can now give a formal definition of ideal measurements.

Definition 10 *A measurement M is called* ideal *if, when it produces an outcome $o \in O(M)$, it moves the state s into the flat $\varphi_{E(o)}(s) = E(o) \wedge (s \vee E(o)^\perp)$.*

We strengthen Axiom 2 the following way:

Axiom 2* *For any ODU $(F_i,\ i \in I)$ there exists an ideal measurement M with $E_M(i) = F_i$.*

In the remaining of the paper we assume that our measurements are ideal.

5.2 Compatible measurements

Informally, two measurements are compatible if performing one of the measurement on a system does not affect the results from performing the other on the same system. For example, all classical measurements are compatible. In order to consider this issue more formally we have to use a notion of commutativity in ortho-lattices.

Definition 11 *Elements x and y of an ortholattice commute (we write x com y) if $x = (x \wedge y) \vee (x \wedge y^\perp)$.*

For example, any central element of an ortholattice commutes with any other element (and the converse assertion is also true). Another simple case is: if $x \leq y$ then x *com* y. In orthomodular lattices the relation of commutativity *com* is symmetric (see [12], Theorem 2 of Section 2). It is known also (see [12], Lemma 3 of Section 3, or [2], Theorem 16.2.4) that in an orthomodular lattice elements x and y commute if and only if $x \wedge (y \vee x^\perp) = x \wedge y$. For this reason we shall suppose further that ortho-lattices are ortho-modular.

Let M and M' be two (ideal) measurements with eigensets $E_M(o)$, $o \in O(M)$, and $E_{M'}(o')$, $o' \in O(M')$).

Definition 12 *The measurements M and M' are* compatible *if every $E_M(o)$ commutes with every $E_{M'}(o')$.*

We assert that compatible measurements are compatible in the previous informal sense, that is performing one of the measurements does not affect the results from the other measurement. Indeed, suppose that a state s is in an eigenset $F := E_M(o)$ and therefore performing M gives outcome o. Suppose further that we perform measurement M' and obtain an outcome o'. Then the state s moves into a state s' which belongs to $G \wedge (s \vee G^\perp)$, where $G = E_{M'}(o')$. All the more, the new state s' belongs to the flat $G \wedge (F \vee G^\perp) = F \vee G$ according to commutativity of F and G. Therefore s' belongs to F, and if we perform the measurement M again we obtain the outcome o.

We assert that two measurements are compatible if and only if they are "coarsening" of a third (finer) measurement.

Definition 13 *A measurement M' is* coarser *than measurement M if any eigenset of M is contained in some eigenset of M'.*

In other words, outcomes of M' can be obtained from outcomes of M by means of a function $f : O(M) \to O(M')$. In this case the eigensets $E_{M'}(o')$ have the form $E_M(f^{-1}(o'))$.

If M' and M'' both are coarsening of M then they are compatible. We give a proof in a particular case (the argument in the general case is similar). Namely, let M be a complete measurement with eigensets $\{x\}$, $\{y\}$ and $\{z\}$, let M' have eigensets $F = x \vee y$ and $\{z\}$, and let M'' have eigensets $\{x\}$ and $G = y \vee z$. We should check that the flats F and G commute. It is clear that $F \wedge G = y$ and $F \wedge G^{\perp} = x$; hence $F = (F \wedge G) \vee (F \wedge G^{\perp})$.

Conversely, let M and M' be two compatible measurements. We construct now a measurement M'' which is a refinement of M and M' simultaneously. To this end we consider the following family of flats $(E_M(o) \wedge E_{M'}(o\prime))$, where $o \in O(M)$, $o' \in O(M^{'})$. It is obvious that all flats in this family are orthogonal to each other. We claim that they form an ODU. To show this we have to check that the join of these flats is equal to $\mathbf{1}$. Here we use the following statement (see [12], Proposition 4 of Section 3):

Lemma 2 *Assume an element a of an orthomodular lattice commutes with elements $b_i, i \in I$. Then a commutes with their join $\vee_i b_i$ and $(\vee_i b_i) \wedge a = \vee_i (b_i \wedge a)$.*

Applying this Lemma to $a = E(o')$ and the family $(b_o = E(o), o \in O(M))$ we get

$$\bigvee_o (E(o) \wedge E(o') = (\bigvee_o E(o)) \wedge E(o') = \mathbf{1} \wedge E(o') = E(o').$$

Therefore the join of all $E(o) \wedge E(o')$ is equal to the join of $E(o')$ that is equal to $\mathbf{1}$. Thus, our family $(E_M(o) \wedge E_{M'}(o')), (o, o') \in O(M) \times O(M'))$ is an ODU. Therefore (due to Axiom 2 or 2*) there exists a measurement with outcome set $O(M) \times O(M')$. Again due to Lemma 2 we see that the measurements M and M' are coarsening of this compound measurement. Thus, we proved the following

Theorem 2 *Measurements M and M' are compatible if and only if there exists a measurement that is a refinement to both M and M'.*

5.3 The Hilbert Space Model

In general, the outcome of an ideal measurement performed on a system in state s does not uniquely determines the resulting state. For instance, take question Q with outcomes F and $F^{\perp}$, we only know that the resulting state will be in the flat $\varphi_F(s) = F \wedge (s \vee F^{\perp})$, where F is the eigenset of the outcome. But if the flat F is not an atom of the lattice $\mathcal{F}(\mathbb{S}, \perp)$, the new state s' can be random.

To see this, let us consider the example of a fly but in 3×2 box. There are two measurements: LR and FCB. Suppose the state F is realized as a probability measure $F(L) = F(C) = F(R) = 1/3$ (of course, $F(F) = 1$ and $F(B) = 0$). If we, in state F, perform a measurement-question with eigensets $\{L\}$ and $\{L\}^{\perp}(= \{C, R\})$ and obtain answer NO, we may conclude that the image of F is not a pure state but a probabilistic "mixture" of C and R. Such a solution is in agreement the Sasaki projection of the state F on the flat $\{C, R\}$ which is equal to $\{C, R\} \wedge (F \vee \{C, R\}^{\perp}) = \{C, R\} \wedge (F \vee L) = \{C, R\} \wedge \mathbf{1} = \{C, R\}$.

This feature is however rather problematic from the point of view of interpretation.[15] To preclude this we introduce a last axiom that assumes that under the impact of a measurement any pure state jumps into another pure state. More precisely, we consider the following

Axiom 3 *For any pure state $s \in \mathbb{S}$ and any flat F the Sasaki image $\varphi_F(s)$ is an atom of the lattice $\mathcal{F}(\mathbb{S}, \perp)$.*

Axiom 3 implies that the state resulting from a measurement performed on a pure state is unambiguously determined. When this axiom is fulfilled, we obtain the mapping $\varphi_F : \mathbb{S} \setminus F^{\perp} \to F$. And a state s, if it jumps into F, jumps precisely in $\varphi_F(s)$.

Suppose now that an orthospace $(\mathbb{S}, \perp)$ is an ortho-separable, orthomodular irreducible space satisfying Axiom 3, and $\mathcal{F}$ is the corresponding ortho-lattice. It is well-known that axiom 3 admits two equivalent formulations. The first is the *covering law*: For any atom a of $\mathcal{F}$ and any flat F the join $a \vee F$ covers F (that is there is no intermediate flat between F and $a \vee F$). And the second is the *exchange property*: Let a and b be two atoms belonging to $\mathbb{S} \setminus F$; if b is a superposition of a and F then a is a superposition of b and F. In particular, if a state a is the superposition of states b and c then c is the superposition of a and b. The irreducibility condition takes the form of "superposition principle": any two different states can form a non-trivial superposition. This means that our lattice $\mathcal{F}$ is submodular (or a matroid). Since $\mathcal{F}$ is an ortholattice (and has a finite height) it is modular.

An irreducible orthospace with modular lattice of flats reminds of a projective geometry. Atoms are called points, flats spanned by two (different) atoms are called lines, flats spanned by three (non-collinear) points are called planes and so on. More precisely, let H be a (finite-dimensional) Euclidean space, as in Section 4.2. And let $\mathbb{S}$ be the set of all one-dimensional vector subspaces of H. The orthogonality is clear. Any flat is given by a vector subspace V and consists of one-dimensional subspaces in V. All the above mentioned properties (or axioms) - atomicity, modularity (consequently, orthomodularity and the covering law) and irreducibility - are fulfilled in this example. The transition probabilities are given as follows: if states s and t have the form $s = \mathbb{R}x$ and $t = \mathbb{R}y$, where x and y are vectors from H, then $s(t) = (x, y)^2/(x, x)(y, y)$, that is $\cos(\varphi)^2$, where φ is the angle between x and y. The Sasaki projection coincides with ordinary orthogonal projection.

In some sense this is not only a special example but a general case. If the height of $\mathcal{F}$ is more than 3 then the lattice $\mathcal{F}$ can be realized as a (ortho)lattice of vector subspaces of some Hermitian space over

[15] It would imply that while we initially had information uniquelly defining the state, the process of measurement results in "knowing less". After the measurement our information does not allow to uniquelly define the state anymore.

some $*$-field K.[16] The details can be found in Beltrametti and Cassinelli (1981) or in Holland (1995).[17] If we additionally require that the orthospace $\mathbb{S}$ is compact and connected (as a topological subspace of $\Delta(\mathbb{S}, \perp)$) then the field K is the real field $\mathbb{R}$, the complex field $\mathbb{C}$ or the field of quaternions $\mathbb{H}$.

Summarizing, under axioms 1-3 (for large enough state space) the non-classical measurement theory developed in this paper coincides (under the additional topological conditions mentioned above) with the Hilbert space formulation of quantum mechanics.

6 Application to Behavioral Sciences

When we turn to behavioral and social sciences, we propose to see an individual as a system (or a collection of irreducible (sub)systems). She is characterized by her state which encapsulates all information about her type (preferences, beliefs, private information). A questionnaire or a decision situation is to be viewed as a measurement device. Actual behavior e.g. the actions taken in a game, the choice made in a decision situation, or the answer given to a questionnaire are measurement outcomes.

First we propose a psychological and behavioral interpretation of some of the axioms and properties of non-classical measurement theory. Thereafter, we provide an example of how this framework can be used to explain a well-documented behavioral "anomaly", i.e., preference reversal. We conclude by suggesting how it could be used to develop a theory of actualized preferences.

6.1 An interpretation of non-classical measurement theory

Ortho-separability

In the classical model all states are pairwise orthogonal (or disconnected). In contrast, in the non-classical model not all (pure) states are orthogonal. Non-orthogonal states are connected with each other in the sense that under the impact of a measurement the state of the system can transit from one state to another. Axiom 1 which secures that the state space is ortho-separable is thus a relaxation of the classical orthogonality assumption.

Features typical of non-classical connectedness are well-known in visual perception analysis. When looking at an ambiguous picture the brain image ("perceptual state") can jump from one (complete) image to another. The different images are mutually exclusive perceptions of one and the same picture yet the state can move between them. In applications to psychology and social sciences an interpretation of non-orthogonal "behavioral states" (type) is as follows. We say that two behavioral states are connected

[16] The case of the height 1 is trivial: $\mathcal{F} = \{0, 1\}$ and the set $\mathbb{P}$ consists of one state. The case of the height 2 is more interest. The (ortho)lattice $\mathcal{F} = \{0, 1\} \cup \mathbb{P}$, and the mapping $s \mapsto s^{\perp}$ acts on the set $\mathbb{P}$ as an involution without fixed points. The case of the height 3 is very intricate and unclear.

[17] Holland (1995) exposes Soler's important result. She formulated a sufficient condition for the generalized Hilbert space corresponding to the quantum logical construction in axiom 1 to 3 to be a Hilbert space in $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Soler′s theorem applies for infinite dimensional space. It appeals to the existence of an infinite orthonormal sequence. Unfortunately, we find no intuition for this condition.

if there exist two (incompatible) decision situations such that when the agent is certain about his attitude in the first, he hesitates in the second and vice-versa. Uncertainty can never be fully sorted out, when it is resolved in one respect, it reappears in another. One way to express it is that a non-classical agent is characterized by an irreducible multiplicity of selves: competing propensities (to act) always coexist in the mind. An illustration is in section 2, example 3. The individual is not be able to resolve her inner conflict in the two situations simultaneously.

The State-property duality

Non-classical measurement theory entails an operational approach to reality that emphasizes interaction with the environment by means of measurement operations. This contrasts with the classical idealistic one. In applications to psychology and social sciences, this approach defines the subject matter of the theory as human behavior in terms of psychological features that can be distinguished by some experimental set-up, i.e., some (first-kind) measurement performed on individuals. More concretely, we would not a priory assume that preferences over any given set of items exists. Instead we would require that an experiment be performed the outcome of which is a full ranking of the items and such that when repeating this experiment with the same person, it yields the same result.

Irreducibility

A (non-trivial) irreducible subsystem is characterized by the fact under the impact of a series of measurements the system can transit from any one pure state to any other. A non-classical system is a structurally "plastic" entity. Moreover the transformations in the state resulting from a measurement cannot be predicted with certainty.

A common theme in psychology and neurobiology is that the individual is a constantly evolving entity. Ledoux (2003) emphasizes what he calls "synaptic plasticity": the self is a constantly changing entity, a network of synapses that is being shaped by experience rather than accumulates experience in a teleological manner (cf. learning). In social sciences, this translates into context dependency (or contextuality), which has been extensively documented in experiments (see for instance Kahneman and Tversky 2000 p. 518-528). The non-classical approach delivers plasticity and contextuality as a consequence of connectedness (non-orthogonality).

In Quantum Physics, the system is plastic, but the "rules of changes" (captured in the correlation matrices that link observables together) do not depend on the individuals systems. Our presumption is that such an assumption is a reasonable approximation when it comes to human beings i.e., there exists sufficient behavioral and psychological regularities. Those regularities are the subject matters of sciences like neurology, psychology and behavioral sciences.

Stability of the state

Axiom 3, is a crucial regularity property which states that under the impact of a measurement a pure state transits to another fully determined pure state. This means that a coarse measurement leaves unperturbed the indeterminacy not sorted out by its set of outcomes.

In the context of psychology an interpretation is that when asked to choose out of an initial state of hesitation, this hesitation is only resolved so as to be able to produce an answer but not more. The remaining indeterminacy is left "untouched". This principle of minimal perturbation thus assumes a certain stability of the behavioral state. If we suppose very reasonably that solving hesitation is connected with some effort, this can be viewed as a principle of economy. In this perspective axiom 3 appears quite reasonable as an approximation when dealing with human systems.

The axiom 1 and 2 and the properties discussed above yield qualitative predictions that can be tested against the predictions of the classical model or those of other behavioral theories. But axiom 3 brings forth so much structure that it brings us very close to the Hilbert space model, which enables making precise quantitative predictions that can be tested empirically**.** It must be emphasized that in quantum mechanics the precise quantitative predictions pertain to *primitive* properties of *simple* systems. In behavioral and social sciences, we are likely more often than less to deal with rather coarse measurements i.e., that capture a complex of type characteristics. This suggests that a mechanic application of the Hilbert space model is not warranted.

Caveat

In applications to behavioral sciences a less attractive feature of our framework is that a measurement erases information about the previous state. We should however recall that a person is expected to be composed of a number of irreducible systems. The loss of memory only applies locally i.e. within one (irreducible) sub-system. Even within such a system, under axiom 3, memory is fully lost only in the case the measurement is complete (not coarse). Yet, our approach implies that to some extent an individual's previous choices are not relevant to her current type. She may recall them but she experiences that she has changed. Implicitly, we assume that at a higher cognitive level, the individual accepts changes in, e.g., her tastes, which are not motivated by new cognition.

6.2 Actualized preferences: an exploration

Classical decision theory assumes that individuals have preferences (a ranking or linear ordering) on an universal set of alternatives X. These preferences may not be known to outsiders. But the individual knows them and they can be revealed by asking questions to the individual (it is assumed that we are dealing with an honest individual). We may ask the individual to tell about his ranking. Or (if we doubt that his behavior is ruled by preferences) we may ask him to make choices from subsets of X. And if his choices satisfy Houthakker's axiom (or Arrow's or Sen's), we eventually reveal the full ranking. How does this simple scheme changes for the case we are dealing with a non-classical or indeterminate individual?

We do not intend to provide a comprehensive answer to this question. Instead we propose a first exploration by ways of discussing a simple example.

Consider the following situation. We have three items a, b, x. We perform an experiment with a population of agents assumed to be identical i.e., in the same state. We first ask them to choose out of the pair (a, b). Thereafter, we ask them to choose out of the set (a, b, x). The following data are observed. Half of the population (to simplify) chooses a out of (a, b) the other half chooses b. In the second experiment (choice out of (a, b, x)), we observe that some of those who previously chose a now choose b and some of those who previously chose b now choose a. This is an instance of "preference reversal" and of violation of "independence of irrelevant alternatives".

We proceed to analyze this situation within our framework. Our assumption is that binary choices are first-kind ideal measurements. We note than an ordering can be obtained by performing a series of binary choices. The first measurement (choice on (a, b)) has two outcomes $a \succ b$ and $b \succ a$. The second measurement (choice out of (a, b, x)) is modelled as a sequence of binary choices. For instance, when confronted with the second choice, the agents would naturally proceed to compare their most preferred option (out of (a, b)) with the new one i.e., they perform measurement (a, x) respectively (b, x). These measurements have two outcomes each $a \succ x$ and $x \succ a$, and $x \succ b$ and $b \succ x$ respectively. After this binary comparison, if the new good was preferred they may want to compare it with the one they first rejected. And they may also repeat the (a, b) comparison. In cases where the own first choice is not confirmed, they may want to repeat the comparison with x. At some point the process stops i.e. they announce a choice. The orthospace is given below

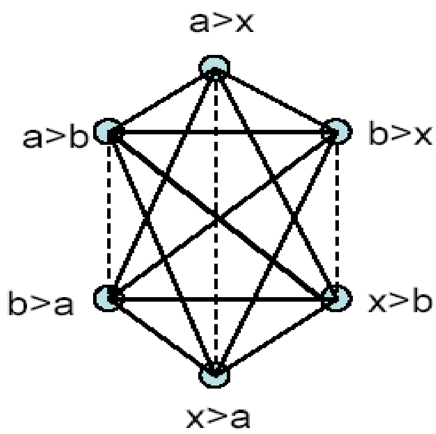

Figure 5

We see that although the behavioral state '$a \succ b$' and '$b \succ a$' are orthogonal, there are pathways from the first to the other for instance through a choice in (a, x). The representation in Figure 5 is not consistent with the existence of an ordering on the whole set (a, b, x).[18] Expressed differently: making a choice out of (a, b, x) is not a first-kind measurement. We say that the set (a, b, x) is not a first-kind choice

[18] If such an ordering existed, the measurement (a, b) and (a, x) would both be coarse measurements of the same property (ordering on (a, b, x). Such measurements are compatible due to the ideality of measurements.

set. The general theory gives no precise guidance as to what happens when performing a measurement that is not first-kind. One possibility is as we did above to assume that it corresponds to performing a sequence of non-commuting measurements each of which is first-kind ideal.[19]

Of course we could have that the preference reversal phenomenon reveals that the person simply lacks any (structured) preferences. But it could also reveal that preferences are non-classical properties. In such a case, we do not assume as in classical decision theory that there exists a single preference order over the universal set of items. Instead, we expect a multiplicity of preference orders defined on incompatible first-kind choice sets. When taking the union of incompatible choice sets, we obtain a set which is not first-kind.

We now attempt to connect our simple example to the experimental literature. A common theme in experimental decision theory is that the process of decision-making in choice is lexicographic and more precisely it involves (the selection of) one prominent attribute. According to decision psychologists, the prominent attribute is selected to facilitate the choice process and therefore must allow to clearly distinguish between the options (cf Kahneman and Tversky p.505). This implies that the composition of the set determines which attribute may qualify as prominent. The issue of prominence has been addressed in connection with violations of “procedural invariance” in a research that started with Lichtenstein and Slovic (1968).[20] This research focuses on binary choices and corresponding pricing or matching tasks. So in particular it does not address issues arising in a situation when there is no single attribute that distinguishes between 3 items simultaneously and that is a reasonable choice criteria. Yet, the prominent hypothesis entails that for some reason the agent cannot produce a synthetic criteria. Our theory may help address this kind of situations.

In each (two-elements) choice set, we can distinguish two candidate prominent attributes each inducing one of the possible choices. They stand in an orthogonality relation with each other. But prominent attributes relevant in different choice sets need not be orthogonal of course. For instance, if in (a, b) “price” (leading to the selection of a) and “the duration of the warranty” (leading to the selection of b) are candidate attributes. We may have that in (a, x) neither price nor warranty are meaningful attributes for guiding the choice. Instead “esthetics” and “size” are relevant. Assume that the agent selected “price” as the prominent attribute (by choosing say a in (a, b)). If we observe the kind preference reversal described above, it could be that some attributes are connected in the sense of non-classical measurement theory. According to that theory when the agent selects e.g. esthetics as a prominent attribute i.e., he chooses a in (a, x) , this impacts on his preferences. His “behavioral state” has been modified which is why he

[19] We note that different sequence lead to different outcomes and that the process needs not converge.

[20] In there first and next following papers (Lichtenstein and Slovi (1971, 1973)), they test the consistency of the orderings revealed in choice versus matching. The experimental fact fact that different elicitation procedures reveal different preference orderings is a theme that lends itself quite immediatly to our approach.

may want to reconsider the (a, b) comparison. When he is in the "esthetics perspective", the value of "the duration of warranty" appears to him in a new light, e.g., as the value of being free from material bother connected with the replacement the item.[21] This may prompt a choice in favor of b.

This example illustrates central distinctions between the classical approach and the non-classical approach to preferences. A non-classical decision-maker does not have preferences over the universal set of items X, only over subsets of X. Preferences over different first-kind (sub)sets are treated as properties some of which may be incompatible i.e., cannot be simultaneously actualized. Similarly to properties, non-classical preference orders defined on incompatible first-kind choice sets are linked by transition probabilities thus implying an essential randomness component. A non-classical agent does not behave as a classical agent because his choice behavior will be characterized by some extent of non-commutativity (consistent with phenomena so different as cognitive dissonance or preference reversal). A non-classical agent does not behave as a random utility maximizer either. In each first-kind set, once he made his choice, he will repeat that choice (unless he is confronted with an incompatible choice set in between).

7 Conclusion

In this paper, we have described the basic structure of non-classical measurement theory (quantum logic). The objective has been to investigate, from a theoretical point of view, whether this framework could be suitable for describing, explaining and predicting human behavior.

We argued that the basic axioms and properties that underline the theory can be given a meaningful interpretation which is consistent with central themes addressed in psychology, behavioral and social sciences. The non-classical man that emerges from our investigation is a structurally plastic creature. This is due to a property of non-orthogonality (connectedness) of behavioral (pure) states (corresponding to psychological properties or type characteristics). This central feature of the theory implies an irreducible uncertainty in behavior, which is also the source of change. When a person interacts with her environment i.e. makes a choice in a given decision context, uncertainty is resolved which prompts a modification of her behavioral state. The non-classical man is essentially contextual.

In a simple decision theoretic example we demonstrate how the framework can be used to explain preference reversal. We conclude by formulating basic ingredients of a theory of actualized preferences based in non-classical measurement theory.

[21] The duration of the warranty is viewed of a signal that the item is not expect to stop functioning before the warranty ends.

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